Modeling and Control of Complex Nonlinear Engineering Systems

Final Exam 2021–2022, Monday 11 April 2022, 8:30 – 10:30

- The exam is **open book**, meaning that the use of the course reader(s) as well as your written notes is allowed. The use of electronic devices is not allowed.
- The exam comprises four problems. Note that Problem 4 comes in **two versions**: please answer the problem according to your educational program (IEM/ME versus AM).
- Different from usual, the exam is for **two hours**.

Problem 1

$(8 + 10 + 4 + 8 = 30$ points$)$

Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u. \tag{1}$$

(a) Characterize the points x_0 for which the system is locally strongly accessible at x_0.

(b) Is the system feedback linearizable to a controllable linear system?

For the remainder of this problem, consider the output

$$y = x_2. \tag{2}$$

(c) What is the relative degree of (1) with output (2)?

(d) Using input-output linearization with output (2), determine a control law that achieves tracking of the reference trajectory

$$y_{\text{ref}}(t) = \cos(\omega t) \text{ for some } \omega > 0.$$

Problem 2

$(5 + 5 = 10$ points$)$

Consider the scalar nonlinear system

$$\dot{x} = x^2 + u. \tag{3}$$

To achieve feedback linearization, one can propose the feedback

$$u = -x^2 - x + v, \tag{4}$$

where v is a virtual input. For $v = 0$, the closed-loop system $\dot{x} = -x$ resulting from (3) and (4) has a globally asymptotically stable equilibrium at $x = 0$.

However, it turns out that, due to modelling errors, the actual nonlinear system reads

$$\dot{x} = (1 + \varepsilon)x^2 + u, \tag{5}$$

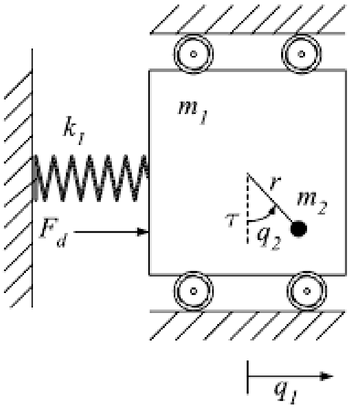
for some small but nonzero ε.

(a) Show that, for the perturbed closed-loop system resulting from (5) and (4), there exist initial conditions x_0 for which the resulting trajectory $x(t)$ grows without bound (i.e., the perturbed closed-loop system does not have a globally asymptotically stable equilibrium).

(b) Is it possible to guarantee stability of the origin for the perturbed closed-loop system by using a linearizing feedback different from (4)? If so, give such feedback. Note that we assume that ε is unknown.

Problem 3 (10 + 10 = 20 points)

Consider a translational oscillator with rotational actuator (TORA) system depicted below.



q_1:	horizontal displacement of the cart
q_2:	angular displacement of the pendulum
m_1:	mass of the cart
m_2:	mass of the pendulum
I:	moment of inertia of m_2
k_1:	spring constant
τ:	input torque
F_d:	external force

The total energy of the system is given by

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^{\mathrm{T}} M(q_2)^{-1}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \frac{1}{2}k_1 q_1^2,$$

with

$$M(q_2) = \begin{bmatrix} m_1 + m_2 & -m_2 r\cos q_2 \\ -m_2 r\cos q_2 & I + m_2 r^2 \end{bmatrix},$$

and where p_1 and p_2 are the translational and rotational momenta variables, respectively. The state space equations of this system can be written in the short-hand form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + u, \end{aligned} \tag{6}$$

with

$$u = \begin{bmatrix} F_d \\ \tau \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

Note that $\dot{q} = M(q_2)^{-1}p$. For reference, the full state space equations are given by

$$\begin{aligned} \dot{q}_1 &= \frac{(I + m_2 r^2)p_1 + (m_2 r\cos q_2)p_2}{(m_1 + m_2)(I + m_2 r^2) - m^2 r^2 \cos^2 q_2} \\ \dot{q}_2 &= \frac{(m_2 r\cos q_2)p_1 + (m_1 + m_2)p_2}{(m_1 + m_2)(I + m_2 r^2) - m^2 r^2 \cos^2 q_2} \\ \dot{p}_1 &= -k_1 x_1 + F_d \\ \dot{p}_2 &= -\frac{\partial}{\partial q_2}\left(\frac{(I + m_2 r^2)p_1^2 + 2(m_2 r\cos q_2)p_1 p_2 + (m_1 + m_2)p_2^2}{(m_1 + m_2)(I + m_2 r^2) - m^2 r^2 \cos^2 q_2}\right) + \tau. \end{aligned}$$

Hint: derivations in the following questions are easier using the short-hand form (6).

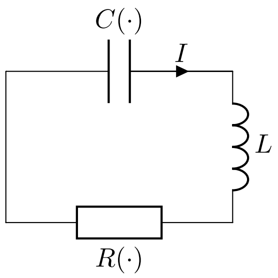
(a) Use the total energy H to determine the passive output corresponding to the input u.

(b) Introduce the feedback control

$$u = -\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} M(q_2)^{-1} p + v,$$

with $d_1 \geq 0$, $d_2 \geq 0$ and where $v = [\, v_1 \; v_2 \,]^{\mathrm{T}}$ is a new input. Is H still a storage function for showing that the input-output pair of (a) is passive for the closed loop system? Please motivate your answer.

Problem 4 (for IEM/ME students) (5 + 7 + 5 + 8 + 5 = 30 points)



Consider the RLC circuit above with nonlinear capacitor and resistor. It can be modelled as

$$\begin{bmatrix} \dot{Q} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} I \\ -L^{-1}C(Q) - L^{-1}R(I) \end{bmatrix}, \tag{7}$$

where Q is the charge of the capacitor and I the current through the network; $L > 0$ is the inductance constant. Moreover, the nonlinear capacitance function $C : \mathbb{R} \to \mathbb{R}$ and resistance function $R : \mathbb{R} \to \mathbb{R}$ are continuous and assumed to satisfy

$$\begin{aligned} QC(Q) &> 0 \qquad \forall\, Q \neq 0, \\ IR(I) &> 0 \qquad \forall\, I \neq 0. \end{aligned}$$

In the remainder of this problem, we will use the notation $x = \begin{bmatrix} Q & I \end{bmatrix}^{\mathrm{T}}$.

(a) Show that $x = 0$ is the unique equilibrium of (7).

As we are interested in stability of the equilibrium $x = 0$, introduce the function

$$V(x) = \tfrac{1}{2} L I^2 + \int_0^Q C(q)\, \mathrm{d}q. \tag{8}$$

(b) Show that V qualifies as a suitable Lyapunov function candidate.

(c) Using V, prove that the equilibrium $x = 0$ is stable (in the sense of Lyapunov).

(d) Is the equilibrium $x = 0$ also asymptotically stable?

(e) Provide a condition (in terms of the functions C and/or R) under which the equilibrium $x = 0$ is *globally* asymptotically stable.

Problem 4 (for AM students) (15 + 15 = 30 points)

Consider N nonlinear systems of the form

$$\dot{x}_i = f_i(x_i, u_i),$$
$$y_i = h_i(x_i),$$

with state $x_i(t) \in \mathbb{R}^{n_i}$ and *scalar* input $u_i(t) \in \mathbb{R}$ and output $y_i(t) \in \mathbb{R}$, with $i = 1, 2, \ldots, N$. In addition, $f_i(0, 0) = 0$ and $h_i(0) = 0$ hold.

For each system i, we assume furthermore that it is zero-state observable and has an L_2-gain bounded by γ_i, i.e., it is dissipative with respect to the supply rate

$$s_i(u_i, y_i) = \tfrac{1}{2}\gamma_i^2\|u_i\|^2 - \tfrac{1}{2}\|y_i\|^2.$$

for some differentiable storage function S_i. We collect states, inputs, outputs, and L_2 gain bounds in the vectors and diagonal matrix

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_N \end{bmatrix},$$

respectively, and assume that the N nonlinear systems are interconnected as

$$u = Ky$$

for some matrix $K \in \mathbb{R}^{N \times N}$.

(a) Assume that there exists a *diagonal* positive definite matrix P such that

$$(\Gamma K)^{\mathrm{T}} P (\Gamma K) - P \prec 0, \tag{9}$$

where $X \prec 0$ indicates that the matrix X is negative definite. Show that the origin $x = 0$ is an asymptotically stable equilibrium point for the interconnected system.

Hint. Write

$$P = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_N \end{bmatrix}$$

and consider the function $V(x) = \sum_{i=1}^{N} p_i S_i(x_i)$.

In the remainder of this problem, we extend the interconnection to include an external input v and external output z as

$$u = Ky + Lv, \quad z = My.$$

(b) Using the same function V as in (a) as a storage function, give a sufficient condition for guaranteeing that the interconnected system is dissipativive with respect to

$$s(v, z) = \tfrac{1}{2}\theta^2\|v\|^2 - \tfrac{1}{2}\|z\|^2,$$

i.e., has an L_2-gain bounded by θ with respect to the external input v and output z. Specifically, give the sufficient condition in terms of a matrix inequality in a similar spirit as (9).

(10 points free)